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MOTIVIC ORTHOGONAL TWO-DIMENSIONAL REPRESENTATIONS OF $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

BY

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ABSTRACT

Given a two-dimensional compatible family of ℓ-adic representations which is motivic and which respects an orthogonal form up to similitudes, we show how to express its L-function in terms of a Hecke character. We give several examples and in particular we analyze a representation associated to a certain $K3$ surface which arose in the study of Kloosterman sums.

Introduction

To compute the ζ-function ζ_V of an algebraic variety V over a number field K one breaks the cohomology groups of V motivically as much as possible. If the pieces are all one-dimensional (example: diagonal hypersurfaces) there is an effective and quite efficient algorithm to perform the computation. For each piece one first writes an explicit, finite list of Hecke characters of type A_0 of K such that the contribution of the piece to ζ_V is a member of the list. This gives a finite list of possibilities for ζ_V. In a second step, one counts the number of points of V over residue fields of K and one compares this to the result coming from each candidate for ζ_V. After sufficiently many primes all possibilities but one for ζ_V are eliminated, and the remaining one is the answer.

On the other hand, if there are higher-dimensional pieces, no general algorithm to compute ζ_V is presently known, although the Langlands program might eventually yield one. In fact in special cases, for example if all the pieces are at most two dimensional and $K = \mathbb{Q}$, there is still a moderately efficient algorithm, in

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which the amount of computation is bounded in advance, but whose success is guaranteed only modulo the Langlands conjectures (see [L]).

In this article we will show that under a certain assumption it is possible to reduce the second, conditional algorithm to the first, unconditional one. The assumption is that the two-dimensional pieces carry symmetric non-degenerate pairings. As a numerical example, we will give an alternative way to [PTV] for computing (the main part of) the zeta function of

$$X = \left\{ (x_1, \ldots, x_5) \in \mathbb{P}^4 | \sum x_i = \sum x_i^{-1} = 0 \right\}$$

which arises from the fifth moment of Kloosterman sums. Notice that in the second step we end up needing only the number of points of X modulo 2, as opposed to considering points modulo primes up to 61 as in [PTV]. We will also discuss similar and other types of examples.

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1. Orthogonal two-dimensional representations

For a number field $K \subset \mathbb{C}$ put $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ and let $C_K = K^\times \setminus \mathbb{A}_K^\times$ be the group of idèle classes of K. Characters of C_K will always be of type A_0 and denoted by greek letters $\mu, \epsilon, \ldots$. We will say such a character is $\mathbb{Q}$-valued if its values on $\mathbb{A}_K^{\times, f}$ are rational. For a place λ of the field of coefficients we will denote the corresponding λ-adic characters by $\mu_\lambda, \epsilon_\lambda, \ldots$. These form compatible systems $\mu_{\mathrm{Gal}}, \epsilon_{\mathrm{Gal}}, \ldots$ of ℓ-adic (or λ-adic) representations of G_K. Let χ: $C_\mathbb{Q} \to \mathbb{C}^\times$ be the cyclotomic character. If ϵ: $C_\mathbb{Q} \to \mathbb{C}^\times$ has finite order we denote by ϵ_{Dir}: $\mathbb{Z}/(\mathrm{cond}\ \epsilon)\mathbb{Z} \to \mathbb{C}$ the corresponding Dirichlet character.

PROPOSITION 1.1: *Suppose* μ: $C_K \to \mathbb{C}^\times$ *is* $\mathbb{Q}$*-valued. Then* $\mu = \left(\chi \circ N_{K/\mathbb{Q}}\right)^k \epsilon$ *for an integer* k *and a quadratic character* ϵ.

Proof: Let $L \supset K$ be finite Galois over $\mathbb{Q}$ and let $l \neq 0$ be an integer such that $\left(\mu \circ N_{L/K}\right)^l = \mu'$ is unramified. Let P be a principal ideal of degree 1 in L above a prime $p \in \mathbb{Q}$ (so $L_P \simeq \mathbb{Q}_p$), let $\pi \in P$ be a generator, and put $\alpha = \mu'_P(\pi)$. Then $\alpha \in \mathbb{Q}^\times$ and by the product formula $\alpha = \prod_{\sigma\colon L \to \mathbb{C}^\times} (\sigma\pi)^{n_\sigma}$ for some integers n_σ. The fractional ideal $(\alpha) \subset \mathbb{Q}$ must be a power of (p), so all the n_σ's are equal. Hence the ∞-type of μ' is composed with the norm $N_{L/\mathbb{Q}}$, so that the ∞-type of μ is composed with $N_{K/\mathbb{Q}}$. There is then an integer k so that $\epsilon = \mu(\chi \circ N_{K/\mathbb{Q}})^{-k}$ is of finite order. Being $\mathbb{Q}$-valued, it is quadratic.

Remark 1.2: If $[K\colon \mathbb{Q}] = 2$ and $\mu\colon C_K \to \mathbb{C}^\times$ has ∞-type not composed with the norm, the method of the preceeding proof shows that the field of values of μ contains K.

In what follows a **motive** means a pure Grothendieck motive satisfying the Weil conjectures and the compatibilities between $H_{\text{ét}}$, H_{deRham}, H_{Betti} and the Hodge filtration. Since Faltings's theory [Fa] is functorial, this means that we also have a Hodge–Tate structure on ℓ-adic cohomology which is compatible with the Hodge type (we only need this for $\ell \gg 0$, when the varieties and correspondences defining M have good reduction). For a character μ of $C_{\mathbb{Q}}$ which is $\mathbb{Q}$-valued the twist $\mathbb{Q}(\mu)$ of the motive $\mathbb{Q}$ by μ is still defined over $\mathbb{Q}$. As usual, $\mathbb{Q}(n) = \mathbb{Q}(\chi^n)$.

THEOREM 1.3: *Let M be a motive of rank 2 over $\mathbb{Q}$ with Hodge numbers $\dim H^{p,q} = \dim H^{q,p} = 1$ where $p > q$. Assume $\langle\ ,\ \rangle\colon M \otimes M \to \mathbb{Q}(\alpha)$ is a non-degenerate symmetric pairing where $\alpha\colon C_{\mathbb{Q}} \to \mathbb{C}^\times$ is $\mathbb{Q}$-valued. Let D be the discriminant of $\langle\ ,\ \rangle$ on $H_{\text{Betti}}(M)$ and let ρ be the associated compatible family of ℓ-adic representations. Then*

1. *D is positive. Putting $K = \mathbb{Q}(\sqrt{-D})$, there is a unique character $\psi\colon C_K \to K^\times \subset \mathbb{C}^\times$ of ∞-type $z \to z^{p-q}$ such that $\rho \otimes K \simeq \chi_{\text{Gal}}^{-q} \otimes \operatorname{Ind}_{G_K}^{G_{\mathbb{Q}}} \psi_{\text{Gal}}$.*
2. *ρ is irreducible, of conductor $\operatorname{cond}\rho = \operatorname{Disc}(K/\mathbb{Q}) \cdot N_{K/\mathbb{Q}}(\operatorname{cond}\psi)$ and determinant $\det\rho = \chi_{\text{Gal}}^{-(p+q)}\epsilon_{\text{Gal}}$, where $\epsilon\colon C_{\mathbb{Q}} \to \mathbb{C}^\times$ is a quadratic character satisfying $\epsilon_{Dir}(-1) = (-1)^{p+q+1}$.*
3. *Let $\sigma\colon K \to K$ be the non-trivial automorphism and put $\psi^\sigma = \psi\sigma$. Then ψ^σ is the complex conjugate $\bar{\psi}$ of ψ and $\psi_{|C_{\mathbb{Q}}} = \alpha$.*
4. *$\det\rho = \alpha_{\text{Gal}}\mu_{\text{Gal}}$, where $\mu\colon C_{\mathbb{Q}} \to \mathbb{C}^\times$ is the quadratic character defining K.*

Proof: For a symmetric non-degenerate pairing $\langle\ ,\ \rangle$ on a finite-dimensional vector space V the group of orthogonal similitudes $\mathrm{GO} = \mathrm{GO}(V, \langle\ ,\ \rangle)$ is the group of maps $g \in \mathrm{GL}(V)$ satisfying $\langle gv, gw\rangle = \lambda(g)\langle v, w\rangle$ for all v, w in V. We view it as an algebraic group. The factor of similitude $\lambda(g)$ satisfies

$\lambda(g)^{\dim V} = (\det g)^2$. When $\dim V$ is even, the connected component of Id_V in GO is $\mathrm{GSO} = \{g \in \mathrm{GO} \mid \lambda(g)^{(\dim V)/2} = \det g\}$. In our case we let GO be the algebraic group over $\mathbb{Q}$ defined by the given pairing $\langle\ ,\ \rangle$ on $H_{\mathrm{Betti}}(M)$. We then have $\alpha_{\mathrm{Gal}}(g)^2 = (\det\rho(g))^2$ for any $g \in G_{\mathbb{Q}}$, and putting $\mu_{\mathrm{Gal}} = \alpha_{\mathrm{Gal}}^{-1}\det\rho$ we see that $\mu_{\mathrm{Gal}}^2 = 1$. Let L be the field cut by μ_{Gal}, so $\operatorname{Ker}\mu_{\mathrm{Gal}} = G_L$, and ρ maps G_L into GSO. Since $\mathrm{GSO} \simeq \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$, it follows that GSO is abelian: in fact $\mathrm{GSO}(\mathbb{Q}) \simeq K^\times$. Therefore $\rho_{|G_L}$ is a rational abelian representation in the sense of Serre [ALR]. It follows that $\rho_{|G_L}$ is locally algebraic, which means that it can be diagonalized: thus

$$\rho_{|G_L} \simeq \begin{pmatrix} \varphi_{\mathrm{Gal}} & 0 \\ 0 & \varphi'_{\mathrm{Gal}} \end{pmatrix}$$

for some characters φ, φ' of C_L.

We claim that the ∞-type of φ, φ' cannot both be composed with the norm $N_{L/\mathbb{Q}}$. If they were we would have $\varphi = (\chi \circ N_{L/\mathbb{Q}})^k\delta$ and $\varphi' = (\chi \circ N_{L/\mathbb{Q}})^{k'}\delta'$ for integers k, k' and Hecke characters δ, δ' of finite orders. By purity we would get $k = k' = (p+q)/2$. But then the Hodge–Tate type of M would be (k,k) and not (p,q), contradicting [Fa]. Therefore $L \neq \mathbb{Q}$, so that L is quadratic. The field of values for φ (or φ') contains L by Remark 1.2 and is contained in K (since ρ is rational), so $K = L$. This proves 4.

The values of φ at Frobenius elements must be Weil numbers in K. They cannot all be in $\mathbb{Q}$, or else $\mathbb{Q}$ would be composed with the norm. Therefore K is quadratic imaginary. By [Fa] and purity again we see that, permuting φ, φ' if necessary, the ∞-type of φ is $z \to z^p\bar{z}^q$. We also see that φ is K-valued and $\varphi' = \varphi^\sigma = \bar\varphi \neq \varphi$. It follows that $\rho \otimes K \simeq \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\varphi_{\mathrm{Gal}}$ and that ρ is irreducible (see [Dur], where the entirely parallel case of dihedral Artin representations is handled). Put $\psi = (\chi \circ N_{K/\mathbb{Q}})^q\varphi$. Then the ∞-type of ψ is $z \to z^{p-q}$ and $\rho \otimes K \simeq \chi_{\mathrm{Gal}}^{-q} \otimes \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\psi_{\mathrm{Gal}}$, proving 1 and the first part of 3.

The formula for $\operatorname{cond}\rho$ is the same as in [Dur]. Since $\det\rho$ is $\mathbb{Q}$-valued it is of the form $\chi_{\mathrm{Gal}}^k\epsilon_{\mathrm{Gal}}$ for ϵ quadratic by Proposition 1.1. By purity (or ∞-type) $k = -(p+q)$. Let $c \in G_{\mathbb{Q}}$ be a complex conjugation. Then $\rho(c)$ interchanges $H^{p,q}(M)$ and $H^{q,p}(M)$, so $\operatorname{Tr}\rho(c) = 0$. Since $c^2 = 1$ it follows that $-1 = \det\rho(c) = \chi_{\mathrm{Gal}}^{-(p+q)}(c)\epsilon_{\mathrm{Gal}}(c) = (-1)^{p+q}\epsilon_{\mathrm{Dir}}(-1)$. This shows 2, and since $\det\rho = \mu_{\mathrm{Gal}}(\psi_{|C_{\mathbb{Q}}})_{\mathrm{Gal}}$ as in [Dur], we obtain $\psi_{|C_{\mathbb{Q}}} = \alpha$, proving the last part of 3 and the theorem. ∎

COROLLARY 1.4:

(1) *The L-function* $L(\rho, s)$ *is* $L(\phi, s) = L(\psi, s+q)$. *In particular it has the usual sort of analytic continuation and functional equation.*

(2) *There exists a unique cuspidal newform* f *such that* $L(\rho, s) = L(f, s+q)$. *This* f *is of weight* $p+1-q$, *level* $\operatorname{cond}\rho$ *and character* $\epsilon = (\alpha\mu\chi^{p+q})_{\mathrm{Dir}}$. *It has complex multiplication by* K.

Example 1.5: Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by $K = \mathbb{Q}(\sqrt{-D})$. The motive $M = H^1(E) \otimes \mathbb{Q}$ satisfies the standard compatibilities between the various types of cohomology. Its Hodge type is $\dim H^{1,0} = \dim H^{0,1} = 1$. Define ϵ as the quadratic character associated to K. The cup pairing $M \otimes M \xrightarrow{\cup} \mathbb{Q}(-1)$ is non-degenerate and alternating, and defining $\langle\ ,\ \rangle\colon M \otimes M \to \mathbb{Q}(-1)(\epsilon)$ by $\langle x, y\rangle = x \cup (\sqrt{-D}\, y)$ gives a non-degenerate symmetric pairing.

The discriminant $\operatorname{Disc}\langle\ ,\ \rangle$ of $\langle\ ,\ \rangle$ is D up to a factor in $(\mathbb{Q}^\times)^2$. To see this choose any $t \in H^1_{\mathrm{Betti}}(E, \mathbb{Q})$. Then $t \cup t = \sqrt{-D}\, t \cup \sqrt{-D}\, t = 0$, and we may view the generator $u = t \cup \sqrt{-D}\, t$ of $H^2_{\mathrm{Betti}}(E, \mathbb{Q}) \simeq \mathbb{Q}$ as a non-zero rational number (in fact we may choose a t for which $u = 1$, but we do not need this). With respect to the basis $t, \sqrt{-D}\, t$ of $H^1_{\mathrm{Betti}}(E, \mathbb{Q})$ we get

$$\begin{aligned}\operatorname{Disc}\langle\ ,\ \rangle &= \det\begin{pmatrix} \langle t,t\rangle & \langle t, \sqrt{-D}\, t\rangle \\ \langle\sqrt{-D}\, t, t\rangle & \langle\sqrt{-D}\, t, \sqrt{-D}\, t\rangle\end{pmatrix} \\ &= \det\begin{pmatrix} t \cup \sqrt{-D}\, t & t \cup (-Dt) \\ \sqrt{-D}\, t \cup \sqrt{-D}\, t & \sqrt{-D}\, t \cup (-Dt)\end{pmatrix} \\ &= \det\begin{pmatrix} u & 0 \\ 0 & Du\end{pmatrix} = Du^2.\end{aligned}$$

As is well-known the corresponding cusp form f has weight 2 and trivial character, reflecting the fact that $p+1-q = 2$ and

$$\det\rho = \alpha_{\mathrm{Gal}}\mu_{\mathrm{Gal}} = (\chi_{\mathrm{Gal}}^{-1}\epsilon_{\mathrm{Gal}})\mu_{\mathrm{Gal}} = \chi_{\mathrm{Gal}}^{-1}.$$

Example 1.6: Let X be a $K3$ surface over $\mathbb{Q}$ with maximal Picard number $\rho = \operatorname{rank} N.S.(X) = 20$. Put $M = (H^2(X)/N.S.(X)) \otimes \mathbb{Q}$, the motive of the transcendental cycles. M satisfies the required compatibilities, and its Hodge type is $\dim H^{2,0} = \dim H^{0,2} = 1$. The cup product pairing $\cup = \langle\ ,\ \rangle\colon M \otimes M \to \mathbb{Q}(-2)$ is non-degenerate and symmetric. Let $c_1, \ldots, c_{20}$ be a basis for $N.S.(X) \otimes \mathbb{Q}$, and let d be the determinant of the matrix of intersection numbers

$c_i \cdot c_j$. By the Hodge index theorem d is negative. Since the discriminant of cup product on $H^2(X)$ is -1 (in fact $\left(H^2(X), \cup\right) \simeq H^3 \oplus \left(-E_8\right)^2$, see [BPV], ch. VIII, proposition 3.2.ii), we get that $D = \mathrm{Disc}(M, \langle\ ,\ \rangle)$ is positive and equal to $-d$ up to a rational square. By the theorem $\det \rho = \alpha_{\mathrm{Gal}} \mu_{\mathrm{Gal}} = \chi_{\mathrm{Gal}}^{-2} \epsilon_{\mathrm{Gal}}$, where $\epsilon = \mu$ and $\epsilon_{\mathrm{Dir}}(p) = \left(\frac{-d}{p}\right)$. This gives a form of weight 3 and odd character ϵ_{Dir}.

Remark 1.7: In Example 1.6 the reference to [Fa] in the proof of the theorem can be replaced by one to [S-I]. They prove that over some extension of $\mathbb{Q}$ the L-function of X is $L(\psi_1^2, s)\ L(\bar{\psi}_1^2, s)$, where ψ_1 is a Hecke character attached to an elliptic curve with complex multiplication. This determines the ∞-type of our ψ to be $z \to z^{-2}$, as required.

Remark 1.8: Serre. In the field K of Theorem 1.3 each ideal class has order dividing $p - q$. Indeed, let P be an ideal of K whose class has order t. Write $P^t = \pi\mathcal{O}$, and let π_P be a uniformizer at P. Then $\psi_P(\pi_P)^t = \psi_P(\pi) = \psi_\infty(\pi)^{-1} = \pi^{p-q}$, which implies $\psi_P(\pi_P)^t \mathcal{O}_K = P^{t(p-q)}$. Hence $P^{p-q} = \psi_P(\pi_P)\mathcal{O}_K$ is principal, so $t | p - q$.

2. The case of $\widetilde{X}$

The minimal projective non-singular model $\tilde{X}$ of the surface X from the introduction is a $K3$ surface of maximal Picard number. The computations in [PTV] determine the parts of $H^*(\widetilde{X})$ which are spanned by algebraic cycles. They show that up to squares the discriminant of the intersection pairing on $N.S.(\widetilde{X})$ is -15 ([PTV], Theorem 2.5). To compute $\zeta_{\widetilde{X}}$ (or ζ_X) it remains to compute the contribution of the transcendental cycles, discussed in Example 1.6. By our Theorem 1.3, $K = \mathbb{Q}(\sqrt{-15})$ and we want to determine ψ, whose ∞-type is $z \to z^{-2}$. Note that our formula for $\det \rho$ checks with [PTV], propositions 3.3.ii and 4.1. The class number of K is 2, and $\mathrm{Cl}(K)$ is generated by $P_2 = (2, \alpha)$ with $\alpha = (1 + \sqrt{-15})/2$. We have $P_2^2 = (\alpha)$. Define an unramified character $\psi_0\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ by $\psi_{0,\infty}(z) = z^{-2}$ and

$$\psi_{0,P}\left(\pi_P\right) = \begin{cases} \pi^2 & \text{if} \quad P = (\pi) \\ \pi^2/\alpha & \text{if} \quad P_2\, P = (\pi) \end{cases}$$

for any ideal P of K and any uniformizer π_P of K at P. Then ψ_0 is well-defined (the only units in $\mathcal{O} = \mathcal{O}_K$ are ± 1), and $\psi_0(K^\times) = 1$. It follows that ψ_0 may be viewed as a Hecke character. Put $\nu = \psi_0^{-1}\, \psi$. Since ν is $K^\times$-valued and of finite order it is quadratic. Since $\tilde{X}$ has good reduction outside 3, 5, our ρ and hence ν are unramified outside primes above 3 and 5. Hence ν factors through

$$C' = K^\times \setminus \mathbb{A}_K^\times / \Big(\big(\prod_{(P,15)=1} \mathcal{O}_P^\times \big) \times \big(\mathcal{O}_{P_3}^\times \times \mathcal{O}_{P_5}^\times\big)^2 \times \mathbb{C}^\times \Big)$$

where P_3, P_5 are the primes of K above 3, 5 respectively. C' is an extension of $\mathrm{Cl}(K)$ by $A = \big(\mathcal{O}_{P_3}^\times \times \mathcal{O}_{P_5}^\times\big) \,/\, \big(\mathcal{O}_{P_3}^\times \times \mathcal{O}_{P_5}^\times\big)^2 \mathcal{O}_K^\times$, where $\mathcal{O}_K^\times$ is embedded diagonally. Hence

$$A \simeq \big(\mathbb{F}_3^\times \times \mathbb{F}_5^\times\big) \,/\, \Big(\big(\mathbb{F}_3^\times \times \mathbb{F}_5^\times\big)^2 \cdot \{\pm(1,1)\} \Big) \simeq \mathbb{F}_5^\times / (\mathbb{F}_5^\times)^2 \simeq \langle \pm 1 \rangle .$$

We claim that C' is cyclic of order 4. Indeed, let $x = (\ldots, x_v, \ldots)$ be the idèle given by $x_v = \sqrt{-15}$ if $v = P_3$ and $x_v = 1$ otherwise. Then $x^2 \neq 1$ in C'. Otherwise the ideal generated by x^2, which is $3\mathcal{O}_K$, would have a generator α which is a square at 5. This is false for both generators ± 3 of $3\mathcal{O}_K$. Hence x has order 4 in C' and C' is cyclic as claimed. It follows that ν factors through $\mathrm{Cl}(K)$.

Now let $\mu_0 \colon K^\times \setminus \mathbb{A}_K^\times \to \mathrm{Cl}(K) \simeq \{\pm 1\}$ be the non-trivial character. Since P_2 is not principal $\mu_0(P_2) = -1$. Set $\rho_1 = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \psi_{0,\mathrm{Gal}}$ and $\rho_2 = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} (\mu_0 \psi_0)_{\mathrm{Gal}}$. Since ν must be 1 or μ_0 we must have $\rho \simeq \rho_1$ or $\rho \simeq \rho_2$. To decide which case occurs we consider what happens at the (good) prime 2. Since 2 splits in K,

$$\mathrm{Tr}\, \rho_1(\mathrm{Frob}_2) = \mathrm{Tr}\Big({\rho_1}_{|_{G_K}} (\mathrm{Frob}_{P_2}) \Big) = \psi_0\,(\mathrm{Frob}_{P_2}) + \overline{\psi_0\,(\mathrm{Frob}_{P_2})} = \alpha + \overline{\alpha} = 1,$$

and likewise $\mathrm{Tr}\,\rho_2(\mathrm{Frob}_2) = -1$. By a direct computation based on counting points modulo 2 on X, one gets $\mathrm{Tr}\,\rho(\mathrm{Frob}_{P_2}) = 1$ (see [PTV], table 1 after lemma 4.4). Hence $\rho \simeq \rho_1$. This checks with [PTV], remark after proposition 4.12. The conductor of ρ is $15 \cdot N_{K|\mathbb{Q}}(1) = 15$, so the corresponding modular form is in $S_3\big(\Gamma_0(15), \big(\frac{-15}{\cdot}\big)\big)$, in accordance with [PTV], Theorem 5.3 and Proposition 5.1.

3. Further examples

Let V_n be the variety $\big\{(x_1, \ldots, x_n) \in \mathbb{P}^{n-1} \,|\, \sum x_i = \sum x_i^3 = 0\big\}$. Let $\mathrm{sgn}\colon S_n \to \{\pm 1\}$ be the signature character of the permutation group on the variables, and put $M = H^{n-3}(V_n)^{\mathrm{sgn}}$. For $n = 7$, 11 and 15 respectively we get rank 2 motives over $\mathbb{Q}$, of Hodge types dim $H^{p,q} =$ dim $H^{q,p} = 1$, where $(p,q) = (3,1), (5,3)$ and $(7,5)$ respectively, and the cup product is symmetric and non-degenerate. These examples give rise to cusp forms of weight 3. The appeal to [Fa] in the proof of Theorem 1.3 appears necessary, especially for $n = 15$.

References

[BPV] W. Barth, C. Peters and A. Van de Ven, *Compact Complex surfaces*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[Fa] G. Faltings, *p-adic Hodge theory*, Journal of the American Mathematical Society **1** (1988), 255–299.

[L] R. Livné, *Cubic exponential sums and Galois representations*, in *Current Trends in Arithmetic Algebraic Geometry*, Contemporary Mathematics, Volume 67, AMS, 1987.

[PTV] C. Peters, J. Top and M. van der Vlugt, *The Hasse zeta function of a $K3$ surface related to the number of words of weight 5 in the Melas codes*, Journal für die reine und angewandte Mathematik **432** (1992), 151–176.

[Dur] J.-P. Serre, *Modular forms of weight one and Galois representations*, in *Algebraic Number Fields* (A. Frohlich, ed.), Academic Press, New York, 1977, pp. 193–297.

[ALR] J.-P. Serre, *Abelian ℓ-Adic Representations and Elliptic Curves*, Benjamin, New York, Amsterdam, 1968.

[S-I] T. Shioda and H. Inose, *On singular $K3$ surfaces*, in *Complex and Algebraic Geometry*, Cambridge University Press, 1977, pp. 117–136.